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Due Date: 2021-09-01

Problem 0 [5 points, Generic Textbook 0.1]

This is a pset template made by Paolo Adajar (paoloadajar@mit.edu) in Summer 2021. I intend to use this template throughout grad school for consistent-looking psets (both for classes I take and classes I am TA for). It includes environments for problems, solutions, and personal reflections.

Problem text be written using the `problem` environment. To display the problem source and number of points, you can use the command `\points[source]{num-points}`; it is demonstrated above. The argument `[source]` is optional. I recommend denoting each problem using `\section*{Problem n \points[source]{num-points}}`.

This template with many pre-installed packages, including:

- `amsmath`, `amsthm`, `amsfonts`, `amssymb`, and `physics` for formatting math,
- `natbib` for citations,
- `graphicx`, `tabularx`, `caption subcaption`, and more for formatting, and
- `version` for excluding problems using `\excludeversion{problem}` (with similar commands for both solutions and reflections).

To use this package, add `\input{paolo-pset.tex}` to the preamble. Additionally, using `\newcommand`, define `\name`, `\email`, `\classname`, `\subject`, `\instructor`, `\assignment`, and `\duedate`. Optionally, define `\collaborators`.

(a) [4 points] This is the text of the first subproblem, which also uses the `problem` environment. The `\points` command can be passed as an optional argument to the `problem` environment to denote the number of points, using the syntax `\problem{\points[source]{num-points}}`. This is recommended for subproblems.

SOLUTION: This is the `solution` environment. It can include both inline math, like $E = mc^2$, and display math text:

$$\sum_{i=1}^{\infty} i = 1 + 2 + 3 + \cdots = -\frac{1}{12}$$

The box that surrounds the `solution` environment will continue across a page break (if needed), as demonstrated with this solution.

Solutions can also use theorems and proofs, following the `amsthm` package, such as

[1] Collaborator(s): Alyssa P. Hacker (`aphacker`), Ben Bitdiddle (`bitdiddle`)

Theorem 1 (Pythagoras). For a right triangle with legs of lengths a and b and hypotenuse of c,

$$a^2 + b^2 = c^2.$$

Proof. Intermediate proofs will end with an empty box (`\square`). □

After intermediate theorems and proofs, you're ready to end your solution. The end of your solution will be marked with a black box. If your solution ends with a `displaymath`, `enumerate`, or `itemize` environment, use `\qedh` to end it with a black box without adding extra space at the end. (This is a modified version of `\qedhere` from `amsthm`). ■

REFLECTIONS: This `reflection` environment, as expected, is used for reflections on solutions. Examples of things to include include:

- Failed solution paths taken
- How the correct solution was found
- What point this question has, pedagogically (and any concepts that were missed)
- Related problems that may be interesting, useful, or cool

The hope is that these reflections will help with my own understanding of this content.

(b) [1 point] Lorem ipsum dolor sit amet, consectetur adipiscing elit. Sed in hendrerit diam. Curabitur quis metus facilisis, consectetur magna nec, dictum nulla. Ut vel lorem magna. Phasellus tristique mauris id leo varius commodo ac eget orci. Etiam ultricies, arcu id accumsan lobortis, mi purus luctus urna, a mattis felis odio sit amet risus. Vivamus suscipit sit amet ante sed volutpat. Sed bibendum egestas porta.

SOLUTION: Suspendisse laoreet ultrices hendrerit. Aenean accumsan ipsum metus, vel venenatis urna volutpat a. Vestibulum feugiat tincidunt metus, id bibendum lectus lacinia interdum. Etiam vitae purus a ante tempus cursus nec non elit. Proin sollicitudin ipsum non tincidunt venenatis. Mauris euismod massa quam, ut volutpat dui pharetra non. Curabitur bibendum a leo nec tristique. Aenean eu aliquam nisi. Proin lobortis nisi non nisi condimentum tempor. Donec a elementum ligula, ut consequat velit. Mauris vitae gravida nisi. Nunc convallis feugiat molestie. Curabitur sed ex hendrerit, tincidunt odio a, tincidunt sapien. ■

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Example Problem: 14.04 PSET 1 #1[2]

(a) Let $X = \mathbb{R}^2_+$ and there be two points $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Suppose $x \succeq y$ if $x_1 > y_1$ or if $x_1 = y_1$ and $x_2 \geq y_2$.

Is the preference relation complete? Transitive? Why or why not?

SOLUTION: This preference relation **is complete**. Note that if $x \succeq y$, we have that $x_1 > y_1$, or that $x_1 = y_1$ and $x_2 \geq y_2$. In the first case, note that $x_1 > y_1$, so $y \not\succeq x$, and thus $x \succ y$. In the second case, if $x_2 > y_2$, then se have that $y \not\succeq x$, and again $x \succ y$. However, if $x_2 = y_2$ and $x_1 = y_1$, then $y \succeq x$ and $x \succeq y$, and so $x \sim y$. In all cases, we have $x \succ y$, $x \prec y$, or $x \sim y$.

This preference relation **is transitive**. Suppose $x \succeq y \succeq z$; then, we have that $x_1 \geq y_1 \geq z_1$. If all inequalities are strict, then $x \succeq z$ and we are done. If both are binding, then we must also have $x_2 \geq y_2 \geq z_2$, and so $x \succeq z$. The remaining two cases where one is strict and one is binding follow similarly.

(Intuitively, one can imagine first comparing the "tens" place and then comparing the "ones" place.) ■

REFLECTIONS: This is the lexicographic preference ordering; I personally think that it is a useful preference ordering to help build intuition behind the different properties of well-behaved preferences.

(I'm going to stop with the "real" reflections here, as I did this PSET about a year ago.)

(b) John has preferences over consumption bundles $(A, B) \in \mathbb{R}^2_+$ characterized by utility function $U(A, B) = A^{\frac{1}{3}} B^{\frac{2}{3}}$. Show that John's preferences satisfy strict monotonicity, local non-satiation, strict convexity, and continuity.

[2] This problem is from Fall 2020; solution is my own.

SOLUTION: We verify all four properties separately.

- **Monotonicity**: Note that at any $X^* = (A^*, B^*)$ with $A^*, B^* > 0$,

$$\left.\frac{\partial U}{\partial A}\right|_{X^*} = \left.\frac{1}{3}\left(\frac{B}{A}\right)^{2/3}\right|_{X^*} > 0$$
$$\left.\frac{\partial U}{\partial B}\right|_{X^*} = \left.\frac{2}{3}\left(\frac{A}{B}\right)^{1/3}\right|_{X^*} > 0$$

  and thus U is monotonic in A and B.

- **Non-satiation**: Note that at any $X^* = (A^*, B^*)$ with $A^*, B^* > 0$, note that

$$U(A^* + \varepsilon, B^*) > U(A^*, B^*)$$

  and so you can always find a better point within ε.

- **Convexity**: Note that the MRS between A and B is given by

$$\frac{\partial U/\partial A}{\partial U/\partial B} = \frac{\frac{1}{3}\left(\frac{B}{A}\right)^{2/3}}{\frac{2}{3}\left(\frac{A}{B}\right)^{1/3}} = \frac{B}{2A}$$

  In addition, we have that

$$\frac{\partial MRS}{\partial A} = -\frac{B}{2A^2} < 0$$

  and so the MRS is decreasing in the quantity of A (and a similar calculation shows that the MRS of B and A is decreasing in B); thus, there are strictly convex indifference curves.

- **Continuity.** Consider a fixed $X^* = (A^*, B^*)$. Note that for X' within an $\delta-$ball around X, we have that

$$U(X') > U(A^* - \delta, B^* - \delta)$$

  Let $\varepsilon = U(A^*, B^*) - U(A^* - \delta, B^* - \delta)$ (this is a finite quantity, as X^* is fixed). As $\delta \to 0$, $\varepsilon \to 0$ because $X' \to X^*$.

  And so, for any $\varepsilon > 0$, we can choose a δ such that all points in the $\delta-$ball around X^* are within ϵ of the utility of X^*. For any $x, y, x \succ y$, choosing an appropriate δ (with $\varepsilon = U(x) - U(y)$) will guarantee that all elements in the $\delta-$ball around x will still be preferred to y. ■

REFLECTIONS: Insert reflection here.

(c) Consider the following constrained maximization problem using the utility function introduced in part (b):

$$\begin{aligned} \max \quad & U(A,B) = A^{\frac{1}{3}}B^{\frac{2}{3}} \\ \text{s.t.} \quad & p_A A + p_B B \leq I \\ & A, B \geq 0 \end{aligned}$$

where $p_A, pB, I > 0$. Let A^*, B^* denote the solution to the above problem.

(i) Can we ever have $A^* = 0$ or $B^* = 0$? Why or why not?

SOLUTION: No, we must have $A^*, B^* > 0$; thus is because for any $I > 0$, choosing A^* or $B^* = 0$ means that $U = 0$, while consuming $\varepsilon > 0$ of each will have positive utility. ■

REFLECTIONS: Insert reflection here. Note that the reflection and solution environment change width for subproblems.

(ii) Can we ever have $p_A A^* + p_B B^* < I$? Why or why not?

SOLUTION: No, we must have $p_A A^* + p_B B^* = I$. Consider some (A, B) such that it costs $< I$. Then, consider $(\overline{A}, B)$ such that $p_A\overline{A} + p_B B = I$; this allocation and has $\overline{A} > A$. Note $U(\overline{A}, B) > U(A, B)$ and so the individual would always strictly prefer this bundle. Thus, the budget must always be fully utilized. ■

REFLECTIONS: Insert reflection here.

(iii) Set up the consumer's Lagrangian and find the first-order conditions. How do you know that these first-order conditions are sufficient to characterize the solution to the consumer's problem? For what values of p_A, p_B will the consumer consume twice as much A as B?

SOLUTION: Expressing this problem using the Lagrangian, we have that

$$\mathscr{L} = A^{\frac{1}{3}}B^{\frac{2}{3}} + \lambda(I - p_A A - p_B B)$$

(We need not incorporate constraints on the non-negativity of each variable because per part (a), the solution is interior. Part (b) tells us the maximizer will be on the budget constraint. Further, because U is quasi-concave and the constraints are convex, we will return the global maximizer.)

Taking derivatives, we have

$$\frac{\partial \mathscr{L}}{\partial A} = \frac{1}{3}\left(\frac{B}{A}\right)^{\frac{2}{3}} - \lambda p_A = 0$$

$$\frac{\partial \mathscr{L}}{\partial B} = \frac{2}{3}\left(\frac{A}{B}\right)^{\frac{1}{3}} - \lambda p_B = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = I - p_A A - p_B B = 0$$

Solving the first and second equations for λ, we get

$$\frac{1}{3}\frac{1}{p_A}\left(\frac{B}{A}\right)^{\frac{2}{3}} = \frac{2}{3}\frac{1}{p_B}\left(\frac{A}{B}\right)^{\frac{1}{3}} \iff \boxed{p_B B = 2p_A A}$$

indicating that the consumer will spend twice their income on B than they will on A.

If we desire $A = 2B$, we must then have that $p_B = 4p_A$. ■

Reflections: Insert reflection here.